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A k-sample trendtest.

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Receptuur

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A k sample trendtest may be used to investigate whether there is an increasing or decreasing trend among k samples. First an example is given of a problem where this test may be applied.

Example: Suppose one wishes to determine whether in mice the duration of the narcosis after the intravenous injection of a new drug is influenced by an increase of the dosage. The duration of the time between the administration of the drug and the awakening of the mice is given in table 1 for four different doses.

TABLE 1. Duration of the narcosis in minutes after the administration of 4 different doses of a drug.

Group i =	1	2	3	4
Dose mg/10 g	1	2	4	8
Duration of the narcosis	17	18	24	54
	6	28	9	24
	17	34	28	14
	32	24	27	7
	15	23	31	40
	7	30	33	79
	38	36	41	80
		51	39	19
		27		48
Mean	18,9	30,1	29,0	38,3

Is it possible to conclude from these data that there is an increasing or decreasing trend of the duration of the narcosis?

The test to be described here is a special case of a series of tests developed by T. J. Terpstra (1955-1956). It is a distributionfree test, i.e. for the applicability no assumptions are necessary on the form of the distribution(s) of the observations. E.g. they need not be normally distributed.

The test may be applied on k independent samples of sizes $n_1, n_2, \ldots, n_k$. The hypothesis to be tested states that the k samples originate from the same population. This hypothesis is rejected if there is a general tendency to increase or decrease among the k samples.

The test statistic of this test will be denoted by V and is calculated from the observations as follows: from k ordered samples $\frac{1}{2}k\ (k - 1)$ pairs of samples (i, j) with $i < j$ may be chosen. For each of these pairs of samples Wilcoxon's W is calculated in the way described by C. van Eeden and Chr. L. Rümke (1958); for the pair (i, j) this statistic is denoted by $W_{i,j}$. In calculating $W_{i,j}$ the observations of the i^{th} sample correspond with the observations of $\underline{x}$ and the observations in the j^{th} sample correspond with the observations of $\underline{y}$. Then $V_{i,j}$ is determined according to (1)

$$(1) \qquad V_{i,j} = \frac{n_i n_j - W_{i,j}}{n_i n_j}.$$

The test statistic V is calculated by summation of all $\frac{1}{2}k\ (k - 1)$ values $V_{i,j}$:

$$(2) \qquad V = \sum_{i<j} V_{i,j}.$$

It will be clear that V assumes in general large positive values if there is an increasing trend and large negative values if there is a decreasing trend.

For large values of $N\ (= n_1 + n_2 + \ldots + n_k)$ the distribution of $\underline{V}$ under the hypothesis tested may be approximated by a normal distribution. This approximation is the better as N increases and as the differences between $n_1, n_2, \ldots, n_k$ decrease.

The mean of this normal distribution is 0, the variance is calculated as follows: for each sample A_i is calculated according to (3) and A is found by summation of the A_i (4).

$$(3) \qquad A_i = \frac{(k + 1 - 2i)^2}{n_i} \qquad (i = 1, 2, \ldots, k)$$

$$(4) \qquad A = \sum_{i=1}^{k} A_i.$$

For all $\frac{1}{2}k\ (k - 1)$ pairs of samples (i, j) with $i < j$ $B_{i,j}$ is calculated according to (5)

$$(5) \qquad B_{i,j} = \frac{1}{n_i n_j}$$

and B is found by summation of all $\frac{1}{2}k\ (k - 1)$ values $B_{i,j}$

$$(6) \qquad B = \sum_{i<j} B_{i,j}.$$

The variance of $\underline{V}$ for the case that no ties are present among the N observations is given in (7)

$$(7) \qquad \sigma^2 = \tfrac{1}{3}\,(A + B).$$

If one wants to test the hypothesis H_0 against the alternative hypothesis of a trend (decreasing or increasing) a twosided critical region is used, consisting of large values of $|V|$. An upper onesided critical region, consisting of large positive values of V, is used if one wants to test H_0 against the alternative hypothesis of an increasing trend; if H_0 is to be tested against the alternative of a decreasing trend a lower critical region, consisting of large negative values of V, is used.

For the two- and onesided tests one calculates respectively

$$\text{(8a)} \qquad u = \frac{|V|}{\sigma} \qquad \text{(twosided)}$$

$$\text{(8b)} \qquad u = \frac{V}{\sigma} \qquad \text{(onesided)}$$

and the tailprobability P may then be found in a table of the standard normal distribution. The hypothesis H_0 is rejected if $P \leqq \alpha$, where α is the level of significance. If H_0 is rejected one concludes that there is an increasing trend if V is positive and that there is a decreasing trend is V is negative.

If ties are present among the N observations σ^2 needs a correction. However if the number and the sizes of the ties are small this correction is in most cases unimportant. Moreover the correction reduces σ^2 and therefore results in a decrease of P; so the correction may be omitted if $P \leqq \alpha$.

The corrected variance is calculated as follows: let h denote the number of ties among the N observations and let $t_1, t_2, \ldots, t_h$ denote their sizes. Let further

$$\text{(9)} \qquad C = \sum_{i=1}^{h} t_i^2$$

$$\text{(10)} \qquad D = \sum_{i=1}^{h} t_i^3$$

then the corrected variance is given by (11):

$$\text{(11)} \quad \sigma^2 = \frac{A\,[n^3 - D - 3(n^2 - C)] - B\,[2(n^3 - D) - 3n\,(n^2 - C)]}{3n\,(n-1)\,(n-2)}.$$

The test will now be applied to the data of table 1. There are $k = 4$ samples; so $\frac{1}{2}.4.3 = 6$ pairs of samples may be formed, i.e. the pairs (1,2), (1,3), (1,4), (2,3), (2,4) and (3,4). For each of these 6 pairs $W_{i,j}$ is calculated (cf. table 2). Then the $V_{i,j}$ are found by means of (1).

TABLE 2. Computation of V and σ^2 for the data of table 1.

pair	$W_{i,j}$		$V_{i,j}$	$B_{i,j}$	
1,2	28		0,56	0,0159	
1,3	28		0,50	0,0179	
1,4	29		0,54	0,0159	
2,3	67		0,07	0,0139	
2,4	69		0,15	0,0123	
3,4	59		0,18	0,0139	
			+	+	
			V = 2,00	B = 0,0898	
i	1	2	3	4	
n_i	7	9	8	9	
A_i	1,2857	0,1111	0,125	1	A = 2,5218

$V = 2{,}00$ is found by summation of these 6 values $V_{i,j}$.

In order to find σ^2 the $B_{i,j}$ are calculated according to (5); by summation of the 6 values $B_{i,j}$ $B = 0{,}0898$ is found. Further for each of the 4 samples A_i is calculated according to (3). Summation of the 4 values gives $A = 2{,}5218$. So (cf. (7))

$$\sigma^2 = \tfrac{1}{3}(2{,}5218 + 0{,}0898) = 0{,}8705$$
$$\sigma = 0{,}9330.$$

Substituting these results in (8a) we obtain $u = 2{,}14$, corresponding to a twosided tailprobability $P = 0{,}032$. So in a twosided test with $\alpha = 0{,}05$ the hypothesis tested is rejected and V being positive we conclude that the duration of the narcosis shows an increasing trend with increasing dose.

As without correction for ties $P < \alpha$ has been found there is no need for such a correction. Moreover the small ties make the correction negligible: the corrected standard deviation is $\sqrt{0{,}8692} = 0{,}9323$.

The use of formula (11) will now be illustrated by means of the example. The number of ties $h = 27$; the observations 7, 17, 27 and 28 each occur twice; 24 occurs three times and the remaining 22 observations each occur once. So

$$t_1 = t_2 = t_3 = t_4 = 2; \; t_5 = 3; \; t_6 = \ldots = t_{27} = 1$$

and (cf. (9) and (10))

$$C = 4 \times 2^2 + 1 \times 3^2 + 22 \times 1^2 = 47$$
$$D = 4 \times 2^3 + 1 \times 3^3 + 22 \times 1^3 = 81.$$

Further $N = 33$, so

$$N^2 = 1089, \qquad N^3 = 35937$$

and

$$N^3 - D - 3N(N^2 - C) = 32730$$
$$2(N^3 - D) - 3N(N^2 - C) = -31446.$$

Substituting these results in (11) we obtain

$$\sigma^2 = \frac{2{,}5218 \times 32730 + 0{,}0898 \times 31446}{3 \times 33 \times 32 \times 31} = 0{,}8692.$$

References

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